BOSTON UNIVERSITY

GRADUATE SCHOOL

Thesis

INTRODUCTION TO THE SOLUTION OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

by

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Introduction

INTRODUCTION

1. Brief History. - The study of Differential Equations began very soon after the discovery of Differential and Integral Calculus, to which it forms a natural sequel. In 1676 Newton solved such an equation by means of infinite series but did not publish his results until 1693, the same year that Liebniz encountered them in his work. From then on progress was rapid. Such men as the Bernoulli brothers, Fontaine, Euler, Clairaut, Taylor and Lagrange became interested, and by the beginning of the nineteenth century many important results had been established but the methods developed were artificial, broad comprehensive principles lacking. Fuch's Memoirs (1866-68) gave the theory of Linear Differential Equations its birth, the Lie Theory of Continuous Groups (1884) advanced it and led Picard and Vessiot to develop a theory which is analogous to the Galois theory for algebraic equations.

2. Importance of Linear Differential Equations. - The problem of solving equations of this type is constantly met in scientific investigations, particularly in those dealing with the study of motions of various kinds. They are encountered in Algebra, Geometry, Mechanics, Physics, Chemistry, and Astronomy. Many problems in tangency, curvature, envelopes, oscillations of mechanical systems and of electric currents, bending of beams, conduction of heat, diffusion of solvents,

velocity of chemical reactions, and celestial mechanics formulate themselves as linear differential equations of one of two types: - those with constant coefficients which are the more important, or those with variable coefficients. In fact, so great is the preponderance of these equations in Physics alone that the study of them in this field might well be called the study of Linear Differential Equations. The frequency of their occurrence in any one of the above mentioned branches of science is sufficient justification for the special attention given them in this paper.

3. Method of Procedure. - The method of reasoning used in developing the subject is largely the inductive type which, I believe, is consistent with the fact that it is of an introductory nature. Absolute rigor has been sacrificed in some instances for the purpose of simplifying the argument.

DEFINITIONS AND ASSUMPTIONS

4. Differential Equations. - An equation involving derivatives or differentials is a differential equation. If the equation involves a single independent variable, it is an ordinary differential equation; otherwise, it is a partial differential equation.

5. Linear Differential Equation. - A differential equation that is of the first degree in the dependent variable and its derivatives is a linear differential equation. The general form of the ordinary linear differential equation is

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4. <u>Differential Equations</u>. - An equation involving derivatives or differentials is a differential equation. If the equation involves a single independent variable, it is an ordinary differential equation; otherwise, it is a partial differential equation.

5. <u>Linear Differential Equation</u>. - A differential equation that is of the first degree in the dependent variable and its derivatives is a linear differential equation. The general form of the ordinary linear differential equation is

$$X_0 \frac{d^n y}{dx^n} + X_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{-------} + X_n y = X$$

Where X_0, X_1, ----- X_n and X are functions of x or constants.

6. Homogeneous - Non-homogeneous Linear Differential Equation. - A linear differential equation in which the right-hand member is zero is a homogeneous linear differential equation. When the right-hand member is a function of x, the equation is said to be non-homogeneous.

7. Solution of Differential Equation. - Any functional relationship (not containing derivatives) between the variables which causes a differential equation to reduce to an identity, is a solution or a particular integral of the equation. The meaning of the term "solution" in this theory is somewhat broader than that given it in Algebra. It is extended to include not only implicit algebraic forms such as $y = \alpha x \sqrt{x^2 + y^2}$, but also symbolic forms such as $y = \int f(x)dx$ which cannot be expressed without integral signs.

8. Assumptions. - It is assumed in this discussion that every linear differential equation of the n^{th} order has a general solution, and that this solution involves n arbitrary independent constants.

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

9. General Theorem. - The general homogeneous linear differential equation is of the form

$$(1)\ X_0 \frac{d^n y}{dx^n} + X_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{-----} + X_n y = 0$$

where X_0, X_1, ----- X_n are constants or functions of x. If $y = y_1$ is a particular integral of this equation, then

$$X_0 \frac{d^n y_1}{dx^n} + X_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \text{------} + X_n y_1 \equiv 0$$

Multiplying this identity through by an arbitrary constant c_1,

$$(2)\ X_0 c_1 \frac{d^n y_1}{dx^n} + X_1 c_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \text{-----} + X_n c_1 y_1 \equiv 0$$

This is the condition that $c_1 y_1$ (=y) is a solution of equation (1). Similarly, if $y = y_2$ is another particular integral of (1), $y = c_2 y_2$ is also a solution, and

$$(3)\ X_0 c_2 \frac{d^n y_2}{dx^n} + X_1 c_2 \frac{d^{n-1} y_2}{dx^{n-1}} + \text{-----} + X_n c_2 y_2 \equiv 0$$

Adding identities (2) and (3)

$$X_0 c_1 \frac{d^n y_1}{dx^n} + X_0 c_2 \frac{d^n y_2}{dx^n} + X_1 c_1 \frac{d^{n-1} y_1}{dx^{n-1}} + X_1 c_2 \frac{d^{n-1} y_2}{dx^{n-1}}$$

$$+ \text{------} + X_n c_2 y_1 + X_n c_2 y_2 \equiv 0$$

This is the condition that $y = c_1 y_1 + c_2 y_2$ is a solution of equation (1). Similarly, if n particular integrals $y = y_1$, $y = y_2$, --- y_n of (1) are known.

(4) $y = c_1y_1 + c_2y_2 + \text{------} + c_ny_n$ is a solution.

Since the general solution of a differential equation involves n independent arbitrary constants, and since $c_1, c_2, \text{---} c_n$ are such constants, (4) is the general solution of equation (1). Hence the theorem:- If $y_1, y_2, \text{----} y_n$ are n independent particular integrals of a homogeneous linear differential equation of the order n, the function $c_1y_1 + c_2y_2 + \text{----} + c_ny_n = y$ is its general solution.

10. Constant Coefficients. Roots of Auxiliary Equation Real and Distinct. - When $X_0, X_1, \text{-----} X_n$ in equation (1) are constants, it may be written

$$(1a)\quad k_0 \frac{d^ny}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + \text{-----} + k_ny = 0$$

Consider the solution of the equation

$$(5)\quad \frac{dy}{dx} + ky = 0$$

which is a special case of (1a).

Separating the variables

$$\frac{dy}{y} + kdx = 0$$

$$\log y + kx = a$$

$$\log y - \log c = -kx$$

$$\frac{y}{c} = e^{-kx}$$

$$y = ce^{-kx}$$

This solution suggests that a particular integral of (1a) may be of the form $y = e^{mx}$. Assuming that this is the case then

$$k_0 m^n e^{mx} + k_1 m^{n-1} e^{mx} + \text{-----} + k_n e^{mx} \equiv 0$$

$$e^{mx}(k_0 m^n + k_1 m^{n-1} + \text{-----} + k_n) \equiv 0$$

If m is a root of

$$(6)\ k_0 m^n + k_1 m^{n-1} + \text{-----} + k_n = 0$$

which is called the auxiliary equation, $y = e^{mx}$ will be a solution. Let m_1, m_2, ----- m_n be distinct real roots of equation (6). By the theorem just proved

$$(7)\ y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \text{-----} + c_n e^{m_n x}$$

will be the general solution of (1a).

The solution of $3\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} - 6y = 0$ will serve as an illustration of the application of (7). The auxiliary equation is

$$3m^2 + 7m - 6 = 0$$

$$(3m-2)(m+3) = 0$$

$$m = \frac{2}{3},\quad m = -3$$

Substituting in (7)

$$y = c_1 e^{\frac{2}{3}x} + c_2 e^{-3x}$$

11. Roots of Auxiliary Equation Complex Imaginary and Distinct. - If all the roots of equation (6) are complex imaginary and distinct, they will occur in pairs of the form $\alpha + \beta i$ and $\alpha - \beta i$. By the theorem ~~proved~~ on page 5 the solution of (1a) will be

$$y = c_1 e^{(\alpha_1 + \beta_1 i)x} + c_2 e^{(\alpha_1 - \beta_1 i)x} + \text{------}$$

$$+ \quad c_{n-1} e^{(\alpha_{\frac{n}{2}} + \beta_{\frac{n}{2}} i)x} + c_n e^{(\alpha_{\frac{n}{2}} - \beta_{\frac{n}{2}} i)x}$$

This can be written

$$y = c_1 e^{\alpha_1 x} e^{\beta_1 ix} + c_2 e^{\alpha_1 x} e^{-\beta_1 ix} + \text{-----} + c_{n-1} e^{\alpha_{\frac{n}{2}} x} e^{\beta_{\frac{n}{2}} ix} +$$

$$c_n e^{\alpha_{\frac{n}{2}} x} e^{-\beta_{\frac{n}{2}} ix}$$

$$(8)\ y = e^{\alpha_1 x}(c_1 e^{\beta_1 ix} + c_2 e^{-\beta_1 ix}) + \text{------------}$$

$$+ e^{\alpha_{\frac{n}{2}} x}(c_{n-1} e^{\beta_{\frac{n}{2}} ix} + c_n e^{-\beta_{\frac{n}{2}} ix})$$

From Euler's Formulas

$$e^{i\theta} = \text{Cos}\,\theta + i \sin\theta$$
$$e^{-i\theta} = \text{Cos}\,\theta - i \sin\theta$$

We may express the exponentials $e^{\beta ix}$ and $e^{-\beta ix}$ in trigonometric form as follows:-

$$e^{i\beta x} = \text{Cos}\,\beta x + i \sin\beta x$$
$$e^{-i\beta x} = \text{Cos}\,\beta x - i \sin\beta x$$

Substituting these values in equation (8)

$$y = e^{\alpha_1 x}(c_1\cos\beta_1 x + c_1 i \sin\beta_1 x + c_2\cos\beta_1 x - c_2 i \sin\beta_2 x)$$

$$+ \text{-----} + e^{\alpha_{\frac{n}{2}} x}(c_{n-1}\cos\beta_{\frac{n}{2}} x + c_{n-1} i \sin\beta_{\frac{n}{2}} x + c_n\cos\beta_{\frac{n}{2}} x$$

$$- c_n i \sin\beta_{\frac{n}{2}} x)$$

$$y = e^{\alpha_1 x}\left[(c_1 + c_2)\cos\beta_1 x + i(c_1 - c_2)\sin\beta_1 x\right] + \text{-----}$$

$$+ e^{\alpha_{\frac{n}{2}} x}\left[(c_{n-1} + c_n)\cos\beta_{\frac{n}{2}} x + i(c_{n-1} - c_n)\sin\beta_{\frac{n}{2}} x\right]$$

If $c_1 + c_2 = A_1$, $i(c_1 - c_2) = B_1$, $c_3 + c_4 = A_2$, $i(c_3 - c_4) = B_2$, and so on,

$$(9)\quad y = e^{\alpha_1 x}(A_1 \cos\beta_1 x + B_1 \sin\beta_1 x) + \text{-----------}$$

$$+ e^{\alpha_{\frac{n}{2}} x}(A_{\frac{n}{2}}\cos\beta_{\frac{n}{2}} x + B_{\frac{n}{2}} \sin\beta_{\frac{n}{2}} x)$$

will be the general solution of (1a) in which, if A_1, A_2, ---, B_1, B_2, ----- are real, c_1, c_2, c_3, ----- must be assumed to be imaginary.

Another convenient form of this solution may be obtained as follows:- Multiply and divide each term of equation (9) by arbitrary constants k_1, k_2, ----- $k_{\frac{n}{2}}$ so chosen that $\frac{A_1}{k_1} = \sin a_1$, $\frac{B_1}{k_1} = \cos a_1$, $\frac{A_2}{k_2} = \sin a_2$, $\frac{B_2}{k_2} = \cos a_2$, and so on. Such a selection is possible for, if $\frac{A_1}{k_1} = \sin a_1$, and $\frac{B_1}{k_1} = \cos a_1$, then $A_1^2 + B_1^2 = k_1^2$ or $k_1 = \sqrt{A_1^2 + B_1^2}$. Since $\sin^2 a_1 + \cos^2 a_1 = 1$, and since

$$\left(\frac{A_1}{\sqrt{A_1^2 + B_1^2}}\right)^2 + \left(\frac{B_1}{\sqrt{A_1^2 + B_1^2}}\right)^2 = 1,$$

either one of these numbers may be taken as the sin a_1 and the other as cos a_1. Similarly for $\frac{A_2}{\sqrt{A_2^2 + B_2^2}}$, $\frac{B_2}{\sqrt{A_2^2 + B_2^2}}$, and so on. Equation (9) can now be written

$$y = k_1 e^{\alpha_1 x}(\sin a_1 \cos \beta_1 x + \cos a_1 \sin \beta_1 x) +$$

$$k_2 e^{\alpha_2 x}(\sin a_2 \cos \beta_2 x + \cos a_2 \sin \beta_2 x) + \text{-----}$$

$$k_{\frac{n}{2}} e^{\alpha_{\frac{n}{2}} x}(\sin a_{\frac{n}{2}} \cos \beta_{\frac{n}{2}} x + \cos a_{\frac{n}{2}} \sin \beta_{\frac{n}{2}} x)$$

$$(10)\quad y = k_1 e^{\alpha_1 x} \sin (\beta_1 x + a_1) + k_2 e^{\alpha_2 x} \sin (\beta_2 x + a_2) + \text{----}$$

$$+ \ k_{\frac{n}{2}} e^{\alpha_{\frac{n}{2}} x} \sin (\beta_{\frac{n}{2}} x + a_{\frac{n}{2}})$$

where k_1, k_2, ----- $k_{\frac{n}{2}}$ and a_1, a_2, ----- $a_{\frac{n}{2}}$ are arbitrary constants.

To illustrate the use of formulas (9) and (10) solve

$$3 \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = 0$$

The auxiliary equation is

$$3m^2 - 2m + 1 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 12}}{6} = \frac{1 \pm \sqrt{2}\, i}{3}$$

Substituting in (9)

$$y = e^{\frac{x}{3}}(A_1 \cos \frac{\sqrt{2}}{3} x + B_1 \sin \frac{\sqrt{2}}{3})$$

Substituting in (10)

$$y = k_1 e^{\frac{x}{3}} \sin (\frac{\sqrt{2}}{3} x + a_1)$$

12. Roots of Auxiliary Equation Multiple. - If equation (6) has a double root m_1, and if the theorem on page 5 holds in this case, the general solution of (1a) will be

$$y = c_1 e^{m_1 x} + c_2 e^{m_1 x} + c_3 e^{m_3 x} + ----- + c_n e^{m_n x}$$

$$y = e^{m_1 x}(c_1 + c_2) + c_3 e^{m_3 x} + ----- + c_n e^{m_n x} .$$

$c_1 + c_2$ is equivalent to a single constant, and consequently only n-1 independent arbitrary constants are involved. This is not the most general solution for n such constants are necessary.

Suppose, that one of the roots $m_2 = m_1 + h$ where h is a small finite quantity. Then

$$(11)\ y = c_1 e^{m_1 x} + c_2 e^{(m_1 + h)x} + c_3 e^{m_3 x} + ----- + c_n e^{m_n x}$$

by the theorem on page 5. Equation (11) can be written

$$(12)\ y = e^{m_1 x}(c_1 + c_2 e^{hx}) + c_3 e^{m_3 x} + ----- + c_n e^{m_n x}$$

Expanding e^{hx} by the exponential series formula

$$e^{hx} = 1 + hx + \frac{h^2x^2}{2!} + \frac{h^3x^3}{3!} + \text{------}$$

Substituting this value in equation (12)

$$y = e^{m_1x}\left[c_1 + c_2(1 + hx + \frac{h^2x^2}{2!} + \frac{h^3x^3}{3!} + \text{-----})\right]$$

$$+ c_3e^{m_3x} + \text{-----} + c_ne^{m_nx}$$

$$(13) \quad y = e^{m_1x}A + Bx + c_2h\frac{h}{2!}x^2 + c_2h\frac{h^2}{3!}x^3 + \text{-----})$$

$$+ c_3e^{m_3x} + \text{-----} + c_ne^{m_nx}$$

As the quantities c_1 and c_2 are arbitrary constants we may choose $c_2 = \frac{B}{h}$ where B is any constant. As h approaches zero c_2h will always equal B and c_2 will become infinitely large. Now if $c_1 + c_2$ is to be an arbitrary constant A, since $c_1 = A - c_2$, c_1 must be chosen infinitely large and negative. Let h approach zero in equation (13) and substitute A for $c_1 + c_2$ and B for c_2h. Then

$$(14) \quad y = e^{m_1x}(A + Bx) + c_3e^{m_3x} + \text{-----} + c_ne^{m_nx}$$

is the general solution of (1a).

Suppose next that three roots m_1, m_2, m_3 of equation (6) are equal. From equation (14) if $m_1 = m_2$

$$(15) \quad y = e^{m_1x}(A + Bx) + c_3e^{m_3x} + \text{-----} + c_ne^{m_nx}$$

Let $m_3 = m_1 + h$. Then substituting in (15)

$$y = e^{m_1x}(A + Bx) + c_3e^{(m_1 + h)x} + \text{-----} + c_ne^{m_nx}$$

$$y = e^{m_1x}\left[(A + Bx) + c_3e^{hx}\right] + c_4e^{m_4x} + \text{-----} + c_ne^{m_nx}$$

Expanding e^{hx} by the exponential series formula

$$y = e^{m_1x}\left[(A + Bx) + c_3 + c_3hx + c_3\frac{h^2x^2}{2!} + c_3\frac{h^3x^3}{3!} + \text{---}\right]$$

$$+ c_4e^{m_4x} + \text{-----} + c_ne^{m_nx}$$

(16) $$y = e^{m_1x}\left[(A + c_3) + (B + c_3h)x + c_3\frac{h^2x^2}{2!} + c_3\frac{h^3x^3}{3!} + \text{-----}\right] + c_4e^{m_4x} + \text{-----} + c_ne^{m_nx}$$

As in the preceding case, choose $c_3 = \frac{A_3}{h^2}$ where A_3 is any constant. As h approaches zero c_3 will always equal A_3 but c_3h and c_3 will become infinitely large. If $B + c_3h$ is to be an arbitrary constant A_2, since $B = A_2 - c_3h$, B must be chosen infinitely large and negative. If $A + c_3$ is to be an arbitrary constant A_1, since $A = A_1 - c_3$, A must be chosen infinitely large and negative. Let h approach zero in (16), and substitute A_1 for $(A + c_3)$. A_2 for $B + c_3h$, and A_3 for c_3h^2. Then

$$y = e^{m_1x}(A_1 + A_2x + A_3x^2) + c_4e^{m_4x} + \text{-----} + c_ne^{m_nx}$$

will be the general solution of (1a).

By a similar line of reasoning it can be shown that if r roots of (6) are equal, the general solution of (1a) will be

$$(17)\ y = e^{m_1 x}(A_1 + A_2 x + A_3 x^2 + \text{-----} + A_r x^{r-1}) \quad +$$

$$c_{r+1} e^{m_{r+1} x} + \text{-----} + c_n e^{m_n x}$$

Thus the difference that an r fold root m_1 produces is that the coefficient of the exponential $e^{m_1 x}$ is no longer an arbitrary constant, but a polynomial of the $(r-1)^{st}$ degree of the form

$$A_1 + A_2 x + A_3 x^2 + \text{-----} + A_r x^{r-1}$$

involving r arbitrary constants.

As an illustration of the application of (17) consider the solution of

$$12 \frac{d^3y}{dx^3} + 4 \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} - 2 = 0$$

The auxiliary equation is

$$12 m^3 + 4 m^2 - 5 m - 2 = 0$$

$$m = -\frac{1}{2}, \quad m = -\frac{1}{2}, \quad m = \frac{2}{3}$$

From equation (17)

$$y = e^{-\frac{1}{2}x}(A_1 + A_2 x) + c_3 e^{\frac{2}{3}x}$$

If the multiple root is complex imaginary, its conjugate will also be multiple of the same order. Suppose m_1 and m_2 of equation (6) are equal and complex imaginary of the form $\alpha + \beta i$, and let m_3 and m_4 be the roots that are the conjugates of m_1 and m_2. These roots then will be of the form $\alpha - \beta i$. By equation (17)

$$y = e^{(\alpha_1 + \beta_1 i)x}(A_1 + A_2x) + e^{(\alpha_1 - \beta_1 i)x}(B_1 + B_2x) + c_5e^{m_5x} + \text{-----} + c_ne^{m_nx}$$

This expression may be written

$$y = e^{\alpha_1 x}.e^{\beta_1 ix}(A_1 + A_2x) + e^{\alpha_1 x}.e^{-\beta_1 ix}(B_1 + B_2x) + c_5e^{m_5x} + \text{-----} + c_ne^{m_nx}$$

$$(18)\quad y = e^{\alpha_1 x}\left[e^{\beta_1 ix}(A_1 + A_2x) + e^{-\beta_1 ix}(B_1 + B_2x)\right] + c_5e^{m_5x} + \text{-----} + c_ne^{m_nx}$$

Substituting the trigonometric forms for $e^{\beta_1 ix}$ and $e^{-\beta_1 ix}$ in (18)

$$y = e^{\alpha_1 x}\left[(\cos\beta_1 x + i\sin\beta_1 x)(A_1 + A_2x) + (\cos\beta_1 x - i\sin\beta_1 x)(B_1 + B_2x)\right] + c_5e^{m_5x} + \text{-----} + c_ne^{m_nx}$$

$$(19)\quad y = e^{\alpha_1 x}\left[(A_1 + B_1 + A_2x + B_2x)\cos\beta_1 x + i(A_1 - B_1 + A_2x - B_2x)\sin\beta_1 x\right] + c_5e^{m_5x} + \text{-----} + c_ne^{m_nx}$$

Let $A_1 + B_1 = A_1'$, $A_2 + B_2 = A_2'$, $i(A_1 - B_1) = A_1''$,

$i(A_2 - B_2) = A_2''$. Substituting in (19)

$$y = e^{\alpha_1 x}\left[(A_1' + A_2'x)\cos\beta_1 x + (A_1'' + A_2''x)\sin\beta_1 x\right]$$

$$+ c_5 e^{m_5 x} + \text{-----} + c_n e^{m_n x}$$

will be the general solution of (1a).

Similarly it can be shown that if a complex imaginary root is r fold, the general solution of (1a) will be

$$(20)\quad y = e^{\alpha_1 x}\Big[(A_1' + A_2'x + A_3'x^2 + \text{-----} + A_r'x^{r-1})\cos\beta_1 x$$

$$+ (A_1'' + A_2''x + A_3''x^2 + \text{-----} + A_r''x^{r-1})\sin\beta_1 x\Big]$$

$$+ c_{2r+1}e^{m_{2r+1}x} + \text{-----} + c_n e^{m_n x}$$

Another form of this result may be obtained as follows:- writing equation (20) in its expanded form and grouping

$$(21)\quad y = e^{\alpha_1 x}\Big[(A_1'\cos\beta_1 x + A_1''\sin\beta_1 x) + x(A_2'\cos\beta_1 x$$

$$+ A_2''\sin\beta_1 x) + \text{-----} + x^{r-1}(A_r'\cos\beta_1 x + A_r''\sin\beta_1 x)\Big]$$

$$+ c_{2r+1}e^{m_{2r+1}x} + \text{-----} + c_n e^{m_n x}$$

Multiply and divide the first group within the bracket by k_1, the second group by k_2, and so on, the k's so chosen that

$\frac{A_1'}{k_1} = \sin a_1$, $\frac{A_2'}{k_2} = \sin a_2$, $\frac{A_2''}{k_2} = \cos a_2$, and similarly for

$\frac{A_1''}{k_1} = \cos. a_1$

the other A's and k's. Substituting in (21)

$$(22)\ y = e^{\alpha_1 x}\left[k_1 \sin(\beta_1 x + a_1) + k_2 x \sin(\beta_1 x + a_2) + \text{-----} + k_r x^{r-1} \sin(\beta_1 x + a_r)\right] + c_{2r+1} e^{m_{2r+1} x} + \text{-----} + c_n e^{m_n x}$$

To illustrate the application of (20) and (22) consider the solution of

$$\frac{d^4 y}{dx^4} + 2\frac{d^3 y}{dx^3} + 3\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 1 = 0$$

The auxiliary equation is

$$m^4 + 2m^3 + 3m^2 + 2m + 1 = 0$$

$$m = \frac{-1 \pm \sqrt{3}\, i}{2}, \quad m = \frac{-1 \pm \sqrt{3}\, i}{2}$$

From equation (20)

$$y = e^{-\frac{1}{2}x}\left[(A_1' + A_2' x)\cos\frac{\sqrt{3}}{2}x + (A_1'' + A_2'' x)\sin\frac{\sqrt{3}}{2}x\right]$$

From equation (22)

$$y = e^{-\frac{1}{2}x}\left[k_1 \sin\left(\frac{\sqrt{3}}{2}x + a_1\right) + k_2 x \sin\left(\frac{\sqrt{3}}{2}x + a_2\right)\right]$$

NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

13. General Theorem. - The general form of the non-homogeneous linear differential equation is

$$(1)\ X_0 \frac{d^n y}{dx^n} + X_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{-----} + X_n y = X$$

where X_0, X_1, ----- X_n and X are functions of x or constants. When the right hand member of this equation is zero, it has been shown that $Y \equiv y = c_1y_1 + c_2y_2 + \text{-----}\ c_ny_n$ is the general solution, and for convenience Y will be referred to as the complementary function of (1). If y_p is a particular integral of (1), then

$$(2)\ X_0 \frac{d^n y_p}{dx^n} + X_1 \frac{d^{n-1} y_p}{dx^{n-1}} + \text{-----} + X_n y_p \equiv X,$$

and since Y is the general solution of (1) when the right hand member is zero, then

$$(3)\ X_0 \frac{d^n Y}{dx^n} + X_1 \frac{d^{n-1} Y}{dx^{n-1}} + \text{-----} + X_n Y \equiv 0$$

Adding the identities (2) and (3)

$$(4)\ X_0 \frac{d^n y_p}{dx^n} + X_1 \frac{d^{n-1} y_p}{dx^{n-1}} + \text{-----} + X_n y_p + X_0 \frac{d^n Y}{dx^n} +$$

$$X_1 \frac{d^{n-1} Y}{dx^{n-1}} + \text{-----} + X_n Y \equiv X$$

This is the condition that $Y + y_p$ is a solution of (1), and since the complementary function contains n independent arbitrary constants, it is the general solution. Hence the theorem:- The general integral of a non-homogeneous linear differential equation is the sum of its complementary function and any particular integral.

14. Symbolic Operator (D-m). - If the coefficients

$X_0, X_1, \text{-----} X_n$ in equation (1) are constants, it can be written

$$(1a)\ k_0 \frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{-----} + k_n y = X,$$

and if we write $\frac{dy}{dx} = Dy$; $\frac{d^2 y}{dx^2} = D^2 y$, ----- $\frac{d^n y}{dx^n} = D^n y$,

equation (1a) becomes

$$(2)\ (k_0 D^n + k_1 D^{n-1} + \text{-----} + k_n)y = X$$

in which the polynomial $k_0 D^n + k_1 D^{n-1} + \text{-----} + k_n$ is a rational integral function of the n^{th} degree by definition, and represents symbolically the differential operator

$$k_0 \frac{d^n}{dx^n} + k_1 \frac{d^{n-1}}{dx^{n-1}} + \text{-----} + k_n .$$

Since any rational integral function of the n^{th} degree with constant coefficients can be written as the product of n linear factors, some alike, all alike, or all distinct, equation (2) becomes symbolically

$$k_0(D-m_1)(D-m_2) \text{ ----- } (D-m_n)y = X$$

By definition of the operator D, $(D-m)y$ means $\frac{dy}{dx} - my$ and $(D-n)y$ means $\frac{dy}{dx} - ny$. The product $(D-m)(D-n)y$ means $(\frac{d}{dx} - m)(\frac{dy}{dx} - ny)$ which equals $\frac{d^2 y}{dx^2} - (m + n)\frac{dy}{dx} + mny$. Symbolically

$$(3)\ (D-m)(D-n)y = \left[D^2 - (m+n) D + mn\right] y$$

That is, the result of operating on y with (D-n) first and then with (D-m) on the result is the same as operating on y with $[D^2 - (m+n)D + mn]$. Moreover, owing to the symmetry of m and n in the right hand member of (3), we see that the order of the operators on the left is not essential. Hence the symbolic representatives of operators of the type here considered behave like algebraic quantities for multiplication; and furthermore, it is obvious that this conclusion will hold regardless of the number of operators involved.

15. Operator Method for Particular Integral. - Suppose we have an equation of the third order, and that it has been divided through by k_0. Symbolically it can be written

(1) $(D-m_1)(D-m_2)(D-m_3)y = X$

Let $(D-m_2)(D-m_3)y = u$, where u is a new function.

Then $(D-m_1)u = X$ or $\frac{du}{dx} - m_1 u = X$.

This is a linear equation of the first order and $e^{-m_1 x}$ is an integrating factor. Hence

$$e^{-m_1 x}\left(\frac{du}{dx} - m_1 u\right) = e^{-m_1 x}X$$

Integrating

$$e^{-m_1 x}u = \int e^{-m_1 x}X\,dx + c$$

$$u = e^{m_1 x}\int e^{-m_1 x}X\,dx + ce^{m_1 x} \quad \text{or}$$

$$(D-m_2)(D-m_3)y = e^{m_1 x}\int e^{-m_1 x}X\,dx + ce^{m_1 x}$$

Let $(D-m_3)y = v$. Then

$$(D-m_2)v = e^{m_1x}\int e^{-m_1x}\,Xdx + ce^{m_1x}$$

This is a linear equation of the first order and e^{-m_2x} is an integrating factor. Therefore

$$ve^{-m_2x} = \int\left[e^{(m_1-m_2)x}\int e^{-m_1x}\,Xdx + ce^{(m_1-m_2)x}\right]dx + c'$$

$$v = e^{m_2x}\int\left[e^{(m_1-m_2)x}\int e^{-m_1x}\,Xdx + ce^{(m_1-m_2)x}\right]dx + c'e^{m_2x}$$

$$(2)\quad v = e^{m_2x}\int e^{(m_1-m_2)x}\int e^{-m_1x}\,Xdxdx + e^{m_2x}\int ce^{(m_1-m_2)x}dx + c'e^{m_2x}$$

$$\int ce^{(m_1-m_2)x}\,dx = \frac{c}{m_1-m_2}\,e^{(m_1-m_2)x}$$

Let $\frac{c}{m_1-m_2} = c''$. Substituting in (2)

$$v = e^{m_2x}\int e^{(m_1-m_2)x}\int e^{-m_1x}\,Xdxdx + e^{m_2x}c''e^{(m_1-m_2)x} + c'e^{m_2x}$$

$$v = e^{m_2x}\int e^{(m_1-m_2)x}\int e^{-m_1x}\,Xdxdx + c''e^{m_1x} + c'e^{m_2x} \quad \text{or}$$

$$(D-m_3)y = e^{m_2x}\int e^{(m_1-m_2)x}\int e^{-m_1x}\,Xdxdx + c''e^{m_1x} + c'e^{m_2x}$$

This is a linear equation of the first order and e^{-m_3x} is an integrating factor, so

$$ye^{-m_3x} = \int\left[e^{(m_2-m_3)x}\int e^{(m_1-m_2)x}\int e^{-m_1x}\,Xdxdx + c''e^{(m_1-m_3)x} + c'e^{(m_2-m_3)x}\right]dx + c_3$$

$$y = e^{m_3x}\int e^{(m_2-m_3)x}\int e^{(m_1-m_2)x}\int e^{-m_1x}\,Xdxdxdx + \frac{c''}{m_1-m_3}\,e^{m_1x} + \frac{c'}{m_2-m_3}\,e^{m_2x} + c_3e^{m_3x}$$

Let $\dfrac{c''}{m_1-m_3} = c_1$, $\dfrac{c'}{m_2-m_3} = c_2$. Then

$$y = e^{m_3x}\int e^{(m_2-m_3)x}\int e^{(m_1-m_2)x}\int e^{-m_1x} X dx dx dx + c_1e^{m_1x} +$$

$$c_2e^{m_2x} + c_3e^{m_3x}$$

is the general solution of (1).

Using the same argument it can be seen that for the n^{th} order equation, the general solution will be

$$(3)\ y = e^{m_nx}\int e^{(m_{n-1}-m_n)x}\int \text{-----} \int e^{(m_1-m_2)x}\int e^{-m_1x} X(dx)^n$$

$$+ c_1e^{m_1x} + c_2e^{m_2x} + \text{-----} + c_ne^{m_nx}$$

in which the first line is the particular integral and the second line the complementary function.

The application of (3) is illustrated by the solution of

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = e^x$$

The auxiliary equation is

$$m^3 - 3m^2 + 2m = 0$$

which has for its ~~first~~ roots

$$m_1 = 2,\ m_2 = 1,\ m_3 = 0$$

The complementary function is

$$Y = c_1e^{2x} + c_2e^x + c_3$$

Substituting in (3)

$$y = e^{0x}\int e^{(1-0)x}\int e^{(2-1)x}\int e^{-2x} e^x dx dx dx + Y$$

$$= \int e^x \int e^x \int e^{-x} dx dx dx + Y$$

$$= \int e^x \int e^x \left[-e^{-x} dx dx \right] + Y$$

$$= -\int e^x \int e^x . e^{-x} dx dx + Y$$

$$= -\int e^x \int dx dx + Y$$

$$= -\int e^x x \, dx + Y$$

$$= - e^x (x-1) + Y \qquad \text{Pierce - 402}$$

$$y = e^x(1-x) + c_1 e^{2x} + c_2 e^x + c_3$$

In the process of integration the constants can be ignored since they give rise only to the terms of the complementary function.

16. Variation of Parameters. - This method of obtaining the particular integral is readily applied, especially if the order of the equation is not high. It consists in considering the constants in the complementary function as undetermined functions of x such that when it is substituted in the given equation we get X, and not zero, as we do when they are constants. Since we have n functions at our disposal and only one condition to impose upon them, we may choose (n-1) other conditions as we please.

Suppose we have a third order equation

$$(1) \quad k_0 \frac{d^3 y}{dx^3} + k_1 \frac{d^2 y}{dx^2} + k_2 \frac{dy}{dx} + k_3 y = X$$

the auxiliary equation of which has distinct roots. Then the complementary function is

$$(2) \quad Y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$$

Differentiating (2)

$$\frac{dY}{dx} = c_1m_1e^{m_1x} + c_2m_2e^{m_2x} + c_3m_3e^{m_3x} + e^{m_1x}\frac{dc_1}{dx} +$$

$$e^{m_2x}\frac{dc_2}{dx} + e^{m_3x}\frac{dc_3}{dx}$$

Using one of the conditions at our disposal, let

$$(3)\quad e^{m_1x}\frac{dc_1}{dx} + e^{m_2x}\frac{dc_2}{dx} + e^{m_3x}\frac{dc_3}{dx} = 0 \quad \text{Then}$$

$$(4)\quad \frac{dY}{dx} = c_1m_1e^{m_1x} + c_2m_2e^{m_2x} + c_3m_3e^{m_3x}$$

Differentiating (4)

$$\frac{d^2Y}{dx} = c_1m_1^2e^{m_1x} + c_2m_2^2e^{m_2x} + c_3m_3^2e^{m_3x} + m_1e^{m_1x}\frac{dc_1}{dx}$$

$$+ m_2e^{m_2x}\frac{dc_2}{dx} + m_3e^{m_3x}\frac{dc_3}{dx}$$

Using another of the conditions at our disposal, let

$$(5)\quad m_1e^{m_1x}\frac{dc_1}{dx} + m_2e^{m_2x}\frac{dc_2}{dx} + m_3e^{m_3x}\frac{dc_3}{dx} = 0. \quad \text{Then}$$

$$(6)\quad \frac{d^2Y}{dx} = c_1m_1^2e^{m_1x} + c_2m_2^2e^{m_2x} + c_3m_3^2e^{m_3x}$$

Differentiating (6)

$$(7)\quad \frac{d^3Y}{dx} = c_1m_1^3e^{m_1x} + c_2m_2^3e^{m_2x} + c_3m_3^3e^{m_3x} +$$

$$m_1^2e^{m_1x}\frac{dc_1}{dx} + m_2^2e^{m_2x}\frac{dc_2}{dx} + m_3^2e^{m_3x}\frac{dc_3}{dx}$$

Substituting (2), (4), (6), and (7) in (1) we get

$$(8)\ k_0 m_1^2 e^{m_1 x} \frac{dc_1}{dx} + k_0 m_2^2 e^{m_2 x} \frac{dc_2}{dx} + k_0 m_3^2 e^{m_3 x} \frac{dc_3}{dx} = X$$

Now (3), (5), and (8) are three linear equations in three unknowns from which $\frac{dc_1}{dx}$, $\frac{dc_2}{dx}$, and $\frac{dc_3}{dx}$ can be found, and c_1, c_2, and c_3 determined by integration. The constants arising from this integration process produce the complementary function again.

It can be seen from this argument that, if an n^{th} order equation were given a system of n equations in n unknowns would result, and c_1, c_2, ----- c_n could be determined.

If the roots of the auxiliary equation are repeated or complex imaginary, the change in the form of the complementary function does not affect the number of constants involved, and consequently causes no difference in this process.

As an illustration of this method, let us solve

$$(9)\ \frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = e^x$$

the equation used to illustrate the method in section 15. The complementary function is

$$(10)\ Y = c_1 e^{2x} + c_2 e^x + c_3$$

Differentiating

$$\frac{dY}{dx} = 2\,c_1 e^{2x} + c_2 e^x + e^{2x}\frac{dc_1}{dx} + e^x \frac{dc_2}{dx} + \frac{dc_3}{dx}$$

Let $e^{2x} \frac{dc_1}{dx} + e^{x} \frac{dc_2}{dx} + \frac{dc_3}{dx} = 0.$ Then

(11) $\frac{dY}{dx} = 2\, c_1 e^{2x} + c_2\, e^{x}$

Differentiating this result

$$\frac{d^2 Y}{dx^2} = 4\, c_1 e^{2x} + c_2 e^{x} + 2\, e^{2x} \frac{dc_1}{dx} + e^{x} \frac{dc_2}{dx}$$. Let

$2\, e^{2x} \frac{dc_1}{dx} + e^{x} \frac{dc_2}{dx} = 0.$ Then

(12) $\frac{d^2 Y}{dx^2} = 4\, c_1 e^{2x} + c_2 e^{x}$

Differentiating this, we get

(13) $\frac{d^3 Y}{dx^3} = 8\, c_1 e^{2x} + c_2 e^{x} + 4\, e^{2x} \frac{dc_1}{dx} + e^{x} \frac{dc_2}{dx}$

Substituting (10), (11), (12), and (13) in (9) we get

(14) $4\, e^{2x} \frac{dc_1}{dx} + e^{x} \frac{dc_2}{dx} = e^{x}$

Solving

$$\begin{cases} e^{2x} \frac{dc_1}{dx} + e^{x} \frac{dc_2}{dx} + \frac{dc_3}{dx} = 0 \\ 2\, e^{2x} \frac{dc_1}{dx} + e^{x} \frac{dc_2}{dx} = 0 \\ 4\, e^{2x} \frac{dc_1}{dx} + e^{x} \frac{dc_2}{dx} = e^{x} \end{cases}$$

$$c_1 = -\frac{1}{2} e^{-x} + C_1$$

$$c_2 = -x + C_2$$

$$c_3 = \frac{1}{2} e^{x} + C_3$$

$$y = (C_1 - \frac{1}{2} e^{-x})e^{2x} + (C_2 - x)e^{x} + (C_3 + \frac{1}{2} e^{x})$$

$$= C_1e^{2x} - \frac{1}{2}e^{x} + C_2e^{x} - xe^{x} + C_3 + \frac{1}{2}e^{x}$$

$$= C_1e^{2x} + (C_2 - x)e^{x} + C_3$$ which is the same result we obtained in section 15 but in slightly different form.

17. Undetermined Coefficients. - This method for finding a particular integral, while not applicable in all cases, is simple when it can be used. It applies to all cases in which the right hand member of the equation contains only terms which have a finite number of distinct derivatives. Such terms are e^{kx}, sin lx, cos mx, and products of these, where k, l, and m are any constants.

The particular integral is found by trial. We are seeking $y = f(x)$ which when substituted in the given equation will reduce the left hand member to the same function that we have in the right hand member. If y occurs in the equation, this leads us to choose the terms of the right hand member, each prefixed by an undetermined multiplier, as a first trial. On substituting this value in the given equation other terms arise from the differentiation. Consequently, we select the terms of the right hand member plus the terms arising from its

differentiation, each prefixed by an undetermined multiplier. Since this quantity is to satisfy the given equation its substitution in it results in an identity. We may then equate the coefficients of like terms to determine the unknown multipliers. This gives a particular integral which when added to the complementary function yields the general solution of the given equation.

Illustrative Example.

Solve $$\frac{d^2y}{dx^2} + y = 2e^x + x^3 - x$$

$$Y = c_1 \cos x + c_2 \sin x$$

$$y_p = ae^x + bx^3 - cx + dx^2 - e$$

$$\frac{dy_p}{dx} = ae^x + 3bx^2 - c + 2dx$$

$$\frac{d^2y_p}{dx^2} = ae^x + 6bx + 2d$$

$$\frac{d^2y_p}{dx^2} + y_p = 2ae^x + bx^3 + (6b - c)x + dx^2 +$$

$$(2d - e) \equiv 2e^x + x^3 - x$$

$a = 1$

$b = 1$

$6b - c = -1$

$c = 7$

$d = 0$

$2d - e = 0$

$e = 0$

Substituting these values in y_p and adding this to the complementary function

$$y = c_1 \cos x + c_2 \sin x + e^x + x^3 - 7x$$

If y is lacking, set X equal to the lowest ordered derivative occurring in the equation and integrate to obtain $y = f(x)$, then proceed as when y is present.

Illustration.

Solve $\dfrac{d^2y}{dx_2} + \dfrac{dy}{dx} = x^2 + \cos x$

$$Y = c_1 + c_2e^{-x}$$

$$\frac{dy}{dx} = x^2 + \cos x$$

$$y = \frac{x^3}{3} + \sin x + c_3$$

$$y_p = ax^3 + bx^2 + cx + d\cos x + e\sin x + c_3$$

$$\frac{dy_p}{dx} = 3ax^2 + 2bx + c - d\sin x + e\cos x$$

$$\frac{d^2y_p}{dx^2} = 6ax + 2b - d\cos x - e\sin x$$

$$\frac{d^2y_p}{dx^2} + \frac{dy_p}{dx} = 3ax^2 + (6a + 2b)x + (2b + c) + (e - d)\cos x - (e + d)\sin x \equiv x^2 + \cos x$$

$$a = \frac{1}{3}$$

$$6a + 2b = 0$$

$$b = -1$$

$$2b + c = 0$$

$$c = 2$$

$$e - d = 1$$

$$-e - d = 0$$

$$d = -\frac{1}{2}$$

$$e = \frac{1}{2}$$

Substituting these values in y_p and adding to Y

$$y = c_1 + c_2 e^{-x} + \frac{1}{3} x^3 - x^2 + 2x - \frac{1}{2}\cos x + \frac{1}{2}\sin x$$

If u is a term of the complementary function and also of the right hand member of

$$(1)\ k_0 \frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{-----} + k_n y = X.$$

the method will fail, for the substitution of u or any of its derivatives for y in it will not give rise to u. From the Theory of Equations we know that if m is an r^{th} order root of a given rational integral equation, then it is an $(r-1)^{th}$ order root of its first derivative, $(r-2)^{th}$ order root of its second derivative and so on to its $(r-1)^{th}$ derivative. If m is a simple root of the auxiliary equation

$$k_0 m^n + k_1 m^{n-1} + \text{-----} + k_n = 0$$

it will not be a root of its first derivative. This amounts to saying that if u, due to m, is a particular integral of

$$(2)\ k_0 \frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{-----} + k_n y = 0,$$

it is not a particular integral of its first derivative. Now if we are to keep u in the discussion, we must use some form of it which on substitution in (1) will give rise to it. This suggests the substitution of xu for y in (1). This results in

$$k_0 x \frac{d^n u}{dx^n} + k_0 n \frac{d^{n-1} u}{dx^{n-1}} + k_1 x \frac{d^{n-1} u}{dx^{n-1}} + k_1(n-1) \frac{d^{n-2} u}{dx^{n-2}}$$

$$+ \text{-----} + k_{n-1} x \frac{du}{dx} + k_{n-1} u + k_n xu = X$$

Grouping

$$x(k_0 \frac{d^n u}{dx^n} + k_1 \frac{d^{n-1} u}{dx^{n-1}} + \text{-----} + k_n u) \quad +$$

$$(k_0 n \frac{d^{n-1} u}{dx^{n-1}} + k_1(n-1) \frac{d^{n-2} u}{dx^{n-2}} + \text{-----} + k_{n-1} u) = X$$

The first parenthesis vanishes but the second, which involves the derivative of (2), does not. Hence we have u and terms arising from it by differentiation and nothing more.

Similarly if u is due to an r fold root, then it is a particular integral of (2) and its first (r-1) derivatives. Hence $x^r u$ is the form we would select in place of u.

To illustrate this principle for a simple root of the auxiliary equation let us solve

$$\frac{d^2y}{dx^2} - y = e^x$$

$$Y = c_1e^x + c_2e^{-x}$$

$$y_p = Axe^x + Be^x$$

$$\frac{dy_p}{dx} = Axe^x + Ae^x + Be^x$$

$$\frac{d^2y_p}{dx^2} = Axe^x + 2\,Ae^x + Be^x$$

$$\frac{d^2y_p}{dx^2} - y_p = 2\,Ae^x \equiv e^x$$

$$A = \frac{1}{2}$$

$$y = c_1e^x + c_2e^{-x} + \frac{1}{2}\,xe^x$$

For an illustration of a multiple root let us solve

$$\frac{d^3y}{dx^3} - 3\,\frac{dy}{dx} - 2\,y = e^{-x}$$

$$Y = c_1e^{-x} + c_2xe^{-x} + c_3e^{2x}$$

$$y_p = Ax^2e^{-x} + Bxe^{-x} + Ce^{-x}$$

$$\frac{dy_p}{dx} = -Ax^2e^{-x} + (2A-B)xe^{-x} + (B-C)e^{-x}$$

$$\frac{d^2y_p}{dx^2} = Ax^2e^{-x} - (4A-B)xe^{-x} + (2A-2B+C)e^{-x}$$

$$\frac{d^3y_p}{dx^3} = -Ax^2e^{-x} + (6A-B)xe^{-x} - (6A-3B+C)e^{-x}$$

$$\frac{d^3y_p}{dx^3} - 3\frac{dy_p}{dx} - 2y = (\cancel{6A}\cancel{-B}\cancel{-2B}\cancel{-6A}\cancel{+3B})xe^{-x}$$

$$-(6A\cancel{-3B}\cancel{+C}\cancel{+3B}\cancel{-3C}\cancel{+2C})e^{-x} = e^{-x}$$

$$A = -\frac{1}{6}$$

$$y = c_1e^{-x} + c_2xe^{-x} + c_3e^{2x} - \frac{1}{6}x^2e^{-x}$$

This method will fail again if a term of the form x^tu, where u is a term of the complementary function, occurs in the right hand member of the given equation. A modification similar to the one used in the preceding discussion applies here. To simplify the argument suppose that we have a third order equation

$$(1)\ k_0\frac{d^3y}{dx^3} + k_1\frac{d^2y}{dx^2} + k_2\frac{dy}{dx} + k_3y = X$$

and that u is due to a simple root of its auxiliary equation. The previous discussion leads us to substitute $x^{t+1}u$ for y in (1). Substituting and grouping we get

$$x^{t+1}\left[k_0\frac{d^3u}{dx^3} + k_1\frac{d^2u}{dx^2} + k_3\frac{du}{dx} + k_3u\right] +$$

$$(t+1)x^t\left[3k_0\frac{d^2u}{dx^2} + 2k_1\frac{du}{dx} + k_2u\right] +$$

$$(t+1)tx^{t-1}\left[3k_0\frac{du}{dx} + k_1u\right] + (t+1)t(t-1)x^{t-2}\left[k_0u\right] = X$$

The first group will vanish, but the others will not, and hence $x^t u$ and terms arising from it by differentiation result, and no more.

It can be readily seen that, if an n^{th} order equation were given, a similar result would be obtained.

If u is due to a double root of the auxiliary equation of (1), the substitution of $x^{t+2}u$ for y is suggested by the preceding multiple root discussion. Substituting and grouping we get

$$x^{t+2}\left[k_0 \frac{d^3u}{dx^3} + k_1 \frac{d^2u}{dx^2} + k_2 \frac{du}{dx} + k_3 u\right] +$$

$$(t+2)x^{t+1}\left[3\,k_0 \frac{d^2u}{dx^2} + 2\,k_1 \frac{du}{dx} + k_2 u\right] +$$

$$(t+2)(t+1)x^t\left[k_0 \frac{du}{dx} + k_1 u\right] + \left[(t+2)(t+1)t x^{t-1} u\right] = X$$

The first two groups vanish, but the others do not. Hence $x^t u$ and terms arising from it by differentiation result, and no more.

Similarly, if we have an n^{th} order equation and if u is due to an r fold root, the substitution of $x^{t+r}u$ for y will give rise to $x^t u$ and terms arising from it by differentiation, and none other.

As an illustration let us solve

$$\frac{d^2y}{dx^2} - 2\,\frac{dy}{dx} + y = x^2 e^x$$

$$Y = c_1e^x + c_2xe^x$$

$$y_p = ax^4e^x + bx^3e^x + cx^2e^x$$

$$= (ax^4 + bx^3 + cx^2)e^x$$

$$\frac{dy_p}{dx} = (ax^4 + bx^3 + cx^2)e^x + (4\,ax^3 + 3\,bx^2 + 2\,cx)e^x$$

$$\frac{d^2y_p}{dx^2} = (ax^4 + bx^3 + cx^2)e^x + 2(4\,ax^3 + 3\,bx^2 + 2\,cx)e^x$$

$$+ (12\,ax^2 + 6\,bx + 2\,c)e^x$$

$$\frac{d^2y_p}{dx^2} - 2\frac{dy_p}{dx} + y_p = 12\,ax^2e^x + 6\,bxe^x + 2\,ce^x \equiv x^2e^x$$

$$12\,a = 1 \qquad b = 0$$

$$a = \frac{1}{12} \qquad c = 0$$

$$y = c_1e^x + c_2xe^x + \frac{1}{12}x^4e^x$$

Summary and Conclusions. - The problem of solving linear differential equations resolves itself into two parts, finding a particular integral and the complementary function.

The finding of the complementary function when the equation has constant coefficients depends wholly upon the solution of an auxiliary equation which is a rational integral function involving a single variable. This solution may become involved if the order of the given equation is high, but otherwise no difficulty is encountered. The form of the

complementary function is exponential or trigonometric according as the roots of the auxiliary equation are real or complex imaginary.

The finding of the particular integral when the equation has constant coefficients may be accomplished by Variation of Parameters, Differential Operator Method, or Undetermined Coefficients. The first two methods have the advantage of absolute generality, but, because they involve integrations, may become laborious when the order of the equation is high. The last method, while not perfectly general, applies in many cases and, as a rule, is to be preferred to the others whenever it can be used because it involves only differentiations and the solution of simultaneous linear algebraic equations. This work also may become long if the order of the equation is high, but usually it is no longer than the other methods, if as long.

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